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Quantum Logic as Partial Infinite-Valued Łukasiewicz Logic

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It is shown that an old and neglected Łukasiewicz (1913) paper contains construction of a many-valued logic which can be in an almost straightforward way used to describe physical experiments. The logic is infinite-valued and its truth values are interpreted as probabilities of experimental confirmation of propositions about results of future experiments. In the case of experiments on quantum objects the logic is partial, i.e. the existence of conjunctions and disjunctions cannot be guaranteed for all pairs of propositions. An outline of previous attempts of using many-valued logics in the description of quantum phenomena is given.

1. MANY-VALUED LOGICS IN QUANTUM MECHANICS. HISTORICAL OUTLINE

The theory of relativity stimulated a growth of interest in non-Euclidean geometries which otherwise would have been treated as mathematical curiosities without links to the real physical world. In a somewhat analogous way quantum mechanics drew attention to non-Aristotelian (nonclassical) logics. However, the necessity of using non-Euclidean geometry in the theory of relativity was clearly visible from its very beginning, while the idea of abandoning classical logic in the description of quantum phenomena was not so obvious. The birth of this idea is usually associated with the paper by Birkhoff and von Neumann (1936) published when quantum mechanics was already a well-established physical theory and, moreover, this idea is still disputed today.

Actually, the first considerations about basing quantum mechanics on many-valued logic were published some years earlier by the Polish logician

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Zygmunt Zawirski (1931, 1932). Zawirski argued that: "... Niels Bohr complementarity theory ... stating that wave and corpuscular points of view in treating all physical phenomena are equivalent, equally admissible, in spite of the fact that these theories are mutually 'excluding' (!) ... can be comprehended only on the basis of three-valued logic, since in bivalent logic the equivalence of a proposition and its negation is necessarily false." He noticed also that: "Heisenberg Uncertainty Principles and statistical character of physical laws forces the usage of infinite-valued logic which has to be taken into account whenever old 'dynamical' laws are replaced by new laws allowing to expect effects with different degrees of probability or, more correctly, with different degrees of possibility."

Unfortunately, neither Zawirski's (1931) paper, which was published in Polish in a local journal, nor its enlarged version (Zawirski, 1932) published in French in a philosophical journal, *Revue de Metaphysique et de Morale*, attracted much attention among physicists. An American astrophysicist, Fritz Zwicky, was more successful: his paper (Zwicky, 1933), published in the *Physical Review*, in which he gave physical arguments against the law of excluded middle and in favor of "many-valuedness of scientific truth," was much more widely discussed (see Jammer, 1974, pp. 345–346). However, the publication of the paper by Birkhoff and von Neumann (1936) soon changed the direction of the attack on the usage of classical logic in quantum mechanics from challenging bivalence to challenging distributivity of the logic underlying quantum phenomena.

The search for the logical basis of quantum phenomena in the domain of many-valued logic was continued from the late 1930s till the early 1950s by Paulette Février (e.g., Février, 1937; Destouches-Février, 1951) and Hans Reichenbach (1944, 1948, 1951, 1952–1953), both using three-valued logic.

Février was undoubtly influenced by epistemological papers of a group of contemporary philosophers (G. Bachelard, F. Gonseth, P. Hertz, L. Rougier) for whom logic was an empirical science which may be changed when new experimental results are obtained. She introduced, besides two ordinary truth values ("true" and "false") ascribed to propositions which, when checked experimentally, yield sometimes true and sometimes false judgements, a third truth value, "absolutely false." This third truth value she ascribed to propositions which can never, by their very nature, be confirmed by an experiment. For example: "the energy E has value E_0" when E_0 does not belong to the energy spectrum. She used two different truth-tables for the conjunction of propositions about quantum systems, depending upon whether propositions were associated with commuting or noncommuting observables. In the first case her truth-table was identical with the truth-table for the conjunction used by Łukasiewicz (1920) in his three-valued logic. In the second case she argued on the basis of the Heisenberg uncertainty relations that the conjunction of propositions associated with noncommuting observables is always "absolutely false."

The best-elaborated and most widely discussed attempt at explaining quantum phenomena on the basis of three-valued logic was published by Hans Reichenbach (1944) in his book, *Philosophic Foundations of Quantum Mechanics*. He distinguished between *phenomena* = microphysical events connected with macroscopic events by "rather short causal chains" and *interphenomena* = interpolations between phenomena without direct manifestation in the form of a macroscopic effect. To illustrate this division, let us consider a typical two-slit experiment, which, according to Feynman's well-known words, contains at its heart the whole mystery of quantum mechanics. Emission of a quantum object by a source and its absorption on a screen are phenomena, while its "path" between the source and the screen, i.e., all of what adherents to the orthodox Copenhagen interpretation forbid even speaking about, belongs to the domain of interphenomena. Reichenbach argued that if someone wants to go beyond the Copenhagen interpretation, i.e., if someone wants to describe interphenomena as well as phenomena believing that they are governed by the same laws of nature, then application of bivalent logic inevitably leads to causal anomalies which vanish when bivalent logic is replaced by three-valued logic. According to Reichenbach the third truth-value *indeterminate* should be treated ontologically and should not be confused with macroscopic epistemological *unknown*.

Logical operations were defined in Reichenbach's three-valued logic independently of one from another, not as in Łukasiewicz' (1920) three-valued logic, where Łukasiewicz began with assuming truth-tables for negation and implication and defined other operations with their aid. Reichenbach considered three types of negation, three types of implication, two types of equivalence, conjunction, and disjunction, out of which one type of negation (called by Reichenbach diametrical), implication, and equivalence (called standard), conjunction, and disjunction were identical with those of Łukasiewicz (1920).

Reichenbach's proposal of using three-valued logic, advocated after his death by Hilary Putnam (1957) evoked much wider discussion than any other attempt at utilizing many-valued logic in the foundations of quantum mechanics, but critical voices prevailed (see Jammer, 1974, pp. 368–375). The same was the attitude of physicists to von Weizsäcker's (1958) "complementarity logic" with complex truth values obtained directly from the mathematical formalism of Hilbert-space quantum mechanics.

Many-valued logics came back to the foundations of quantum mechanics through the back door together with fuzzy set theory. Initially this return remained unnoticed, although it was clear nearly from the very birth of fuzzy set theory that infinite-valued Łukasiewicz logic is related to fuzzy sets in the same way as bivalent logic is related to traditional ("crisp") sets (Giles, 1976). In the late 1980s a group of Slovak mathematicians and, independently, the present author noticed the remarkable similarity of some families of fuzzy sets to quantum logics in the Birkhoff–von Neumann sense, i.e., to orthomodular posets or lattices (Pykacz, 1987a,b; Riečan, 1988; Dvurečenskij and Riečan, 1988; Dvurečenskij and Chovanec, 1988). This observation began a continuous flow of papers, the number of which exceeds now 150 [unfortunately, published mostly in local and/or mathematical journals; see Pykacz (1992, 1993) and Cattaneo *et al.* (n.d.) for bibliographical references] and which can be treated as belonging to the domain of "fuzzy quantum logic."

Families of fuzzy sets can be endowed with various sums and products implied by various connectives of infinite-valued Łukasiewicz logic (Giles, 1976). Some of these operations, like the most frequently used original Zadeh (1965) ones, are distributive and, therefore, are not very well suited for description of quantum systems. However, operations on fuzzy sets called by Giles (1976) *bold intersection* and *bold union*, the logical roots of which can be also traced back to Łukasiewicz (Giles, 1976; Pykacz, 1992; cf. also Frink, 1938), are not only nondistributive, but also satisfy a formal counterpart of the orthomodular identity (Pykacz, 1994). Moreover, these operations allow one to define "fuzzy quantum logic" which is an orthomodular poset, i.e., a quantum logic in the Birkhoff–von Neumann sense. This definition can be further translated into the language of infinite-valued Łukasiewicz logic (Pykacz, 1994, n.d.-a). Therefore, these last results combine two features which, according to various authors, should distinguish the logic that underlies quantum mechanics from classical logic: many-valuedness advocated by Zawirski already in 1931 and nondistributivity dominating in the quantum logic approach since Birkhoff and von Neumann.

2. LOGICAL FOUNDATIONS OF EXPERIMENTAL THEORIES

2.1. Introduction

In 1913 Jan Łukasiewicz published a paper the English title of which, "Logical foundations of probability theory," is paraphrased as the title to the present section. In this paper [N.B.: published 20 years before Kolmogorov's (1933) fundamental textbook on probability theory] his aim was to clarify the notion of probability which was at that time still alien to the rest of "well-established" mathematics by reducing it to the notions of many-valued logic. In particular, he defined the *probability* that a proposition would turn out to be true as its *truth-value*, while nowadays, since probability theory belongs to the mathematical canon while many-valued logic still does not, it is more natural to reverse this order. This is actually done in the present paper.

Let us also notice that Łukasiewicz (1913), although aimed at clarifying the foundations of probability theory, is actually his first paper on many-valued logics. This falsifies the widespread belief that Łukasiewicz began his studies of many-valued logics in the early 1920s (Łukasiewicz, 1920). However, Łukasiewicz (1913) differs a lot from his later papers on many-valued logics, which are distinguished by their precision and mathematical elegance to the extent that, especially in his later papers (cf. Łukasiewicz, 1970) their main parts boil down to strings of formulas. This "early Łukasiewicz" paper is much less precise and transparent. In this paper intuitive arguments are sometimes used instead of formal ones, formulas valid in bivalent logic but not necessarily valid in many-valued logic are sometimes assumed without checking, and definitions of some important notions have to be extracted from the context. Nevertheless, Łukasiewicz (1913), despite its flaws, or maybe even because of its flaws, since they make its language more similar to the language used by experimentalists, contains ideas which can be in a nearly straightforward way applied in the description of any experiment. To show this I shall maintain as far as possible the formal similarity of the rest of this section to the most important first part of Łukasiewicz (1913) by keeping its order and argumentation. I shall use, whenever possible, Łukasiewicz' original formulations (his exact words written in italic) adopting them to my aim: statements about results of experiments. In the case of necessary deviations from Łukasiewicz' original formulations his exact words will be quoted inside square brackets. Quotations are taken from the English translation of Łukasiewicz (1913) contained in the collection of his selected papers (Łukasiewicz, 1970) and will be preceded by page numbers referring to this book.

2.2. Indefinite Propositions

(p. 16) *I call indefinite those propositions which contain* statements about results of not-yet-performed experiments [*a variable*]. *For instance,* "detector will click in the next run of an experiment," "Schrödinger's cat will be found alive 1 hour after closing the box" [*"x is an Englishman," "x is greater than 4"*].

I shall consider hereafter only those indefinite propositions for which there exist well-defined experimental procedures allowing one to check them [*in which the values of the variables range over a well-defined finite class of individuals*].

If a run of an experiment is completed [*in an indefinite proposition we substitute the variable for one of its values*] *we obtain a definite singular judgement which is either true or false. For instance* "detector clicked," "Schrödinger's cat was found alive 1 hour after closing the box"[2] [*"5 is greater than 4," "3 is greater than 4"*].

2.3. Truth Values

(p. 17) *By the truth value of an indefinite proposition I mean the* probability that it will yield, when checked experimentally, a true singular judgement [*ratio between the number of values of the variables for which the proposition yields true judgements and the total number of values of the variables*].

It is clear that the original Łukasiewicz definition is actually identical with the "classical definition of probability" if one assumes that all values of the variables are equiprobable. In the year 1913 Łukasiewicz could not have used Kolmogorov's (1933) abstract definition of probability; moreover, his aim was to explain the notion of probability by logical notions, not vice versa. Therefore, only on p. 48, after long considerations, does he come to the conclusion, which we can adopt literally:

(p. 48) *The degree of probability of an indefinite proposition is identical with its truth value.*

Nowadays we are fully acquainted with the notion of probability. Moreover, the theory whose logical foundations are the main objective of this paper, quantum mechanics, seems to be probabilistic by its very nature. Therefore, I do not assume that in order to determine truth values of indefinite propositions one really has to repeat experiments many times, calculate frequencies of results, and adopt them as truth values (believing that frequencies converge to probabilities when the numbers of runs go to infinity). On the contrary, I assume the full freedom in using predictive powers of the best contemporary theories to calculate probabilities = truth values of indefinite propositions, since by the words "experimental theories" used in the title of this section I mean theories whose predictions can be, at least in principle, verified experimentally.

2.4. Implication

(p. 17) *The relation of implication, or the relation between reason and consequence, holds between two indefinite propositions a and b if for every pair of* runs of experiments designed to check *a* and *b* [*values of the variables occurring in a and in b*] *either the reason a yields a false judgement or the consequence b yields a true judgement.*

Let us note that the traditional (Birkhoff–von Neumann type) quantum logical "implication," usually identified with partial order, is also a relation

[2] Future tense used for indefinite (experimental) propositions and past tense for definite singular judgements help to distinguish them and make the whole issue more clear. Thanks are due to Dirk Aerts for reassuring me on this idea.

which holds between some pairs of propositions, not an operation which could be performed on an arbitrary pair of them. However, the relation of implication defined above concerns results of individual runs of experiments. Therefore, it should be "stronger" than the partial order relation of Mackey (1963) type defined on a traditional quantum logic in a "statistical" way, i.e., with the aid of a probability measure. I shall show in the sequel that this is really the case.

We can distinguish, following Łukasiewicz, three cases when the relation of implication between indefinite propositions *a* and *b* holds: Two obvious cases when either the reason *a* yields false judgements or the consequence *b* yields true judgements in every run of an experiment designed to check it (the results of an experiment which checks the other proposition are in these two cases irrelevant), and the third case when:

(p. 17) *Neither the reason a yields false judgements* in every run of an experiment designed to check it [*for all values of its variables*] *nor the consequence b yields true judgements* in every run of an experiment designed to check it [*for all values of its variables*]. *Then the statements a and b must* be checked in the same experiment [*contain the same variable x*] *and all* runs of this experiment [*values of x*] *which verify the reason a must verify the consequence b.*

Łukasiewicz argued that if *a* and *b* contained different variables, then one could always choose independently their values in such a way that *a* would yield a true singular judgement and *b* a false singular judgement, contrary to the adopted definition. The same considerations can be applied to our "experimental" case: if *a* and *b* were checked in two independent experiments with uncorrelated results, then we could combine their results at will to find a pair of results for *a* and *b* which would not fulfill the conditions of the adopted definition of the relation of implication.

Łukasiewicz' relation of implication establishes a deductive link between singular judgements obtained from its precedent and consequent, which was stressed by Łukasiewicz by calling it *the relation between reason and consequence*. If *a* is a reason for *b*, i.e., if the relation of Łukasiewicz implication between *a* and *b* holds, then from the truth of any singular judgement obtained from *a* we deduce that a singular judgement "simultaneously obtained" from *b* is necessarily true. However, this link does not have to be causal (although it can be), which is shown by the following example.

Example 1. Let an experimental arrangement consist of a source which emits spin-$^1/_2$ particles and two perpendicularly oriented Stern–Gerlach (SG) apparata in such a configuration that only those particles which leave the first SG apparatus by its "up" channel enter the second SG apparatus. Let a_{up} (a_{down}) be the indefinite proposition "a particle will leave the *second* SG apparatus by its 'up' (resp. 'down') channel" and b_{up} be the proposition "a particle will leave the *first* SG apparatus by its 'up' channel." Obviously according to the Łukasiewicz definition both a_{up} and a_{down} imply b_{up}, since in no run of an experiment can a particle be found in the "up" or "down" channel of the second SG apparatus without leaving the first SG apparatus by its "up" channel. However, leaving the *second* SG apparatus by any of its outgoing channels is obviously not a cause for (actually, in the given experimental arrangement it is a result of) previously leaving the *first* SG apparatus by its "up" channel.

This example shows also that the relation of Łukasiewicz implication can hold between two indefinite propositions predicting the results of two experiments on the same quantum object even if observables measured in these experiments do not commute.

2.5. Theorem on the Truth Value of a Reason

(p. 19) *The truth value of a reason cannot be greater than the truth value of the consequence.*

If we adopt symbols used in Łukasiewicz (1913), $a < b$ to denote that the relation of implication holds between indefinite propositions a and b, and $w(a)$ to denote the truth value of a proposition a, then this theorem can be written symbolically:

$$(a < b) \Rightarrow [w(a) \leq w(b)] \qquad (1)$$

where $\Rightarrow$ denotes ordinary bivalent implication, since bivalent logic is obviously a metalogic for the infinite-valued logic of indefinite propositions.

Łukasiewicz proved this theorem by counting numbers of values of the variables for which a reason a and a consequence b yielded true singular judgements. He could do this since he assumed these numbers to be finite, but we can also do it if we believe that frequencies converge to probabilities when the numbers of runs of experiments go to infinity, which is a general belief among physicists.

In general, theorem (1) cannot be reversed: if indefinite propositions a and b are checked in two independent experiments the results of which are not correlated, then "statistical" inequality $w(a) \leq w(b)$ says nothing about results of particular runs of these experiments. Therefore, it may happen that in some pairs of runs of these experiments a would yield a true and b a false singular judgement, so a would not be a reason for b. This justifies the claim that the Łukasiewicz relation of implication is stronger than the partial order relation of Mackey type on a traditional quantum logic: if the formula (1) is valid in any state of a physical system, then it establishes partial order of Mackey type between indefinite propositions a and b. Therefore, in our approach the link between the relations of partial order and implication is opposite to the quantum logic tradition originated by Husimi (1937): the partial order relation does not define, but is defined by the relation of implication between propositions.

Łukasiewicz justified the following statement, which he called *theorem on the truth value of a reason*, again by counting in a specific example numbers of values of the variables which satisfy the reason a and the consequence b. Actually this statement is not a theorem, but an axiom of his calculus, which he explicitly stated a little bit later.

(p. 20) *The truth value of a reason, augmented by the truth value of the logical product of the negation of the reason and of the consequence, equals the truth value of the consequence.*

2.6. Negation, Logical Product, Logical Sum, and Equivalence

In the *theorem on the truth value of a reason* Łukasiewicz introduced, in passing, two new important notions: *negation* and *logical product* of indefinite propositions. He gave no explicit definitions of these notions; however, it can be inferred from the context that by the *negation* of an indefinite proposition a he meant an indefinite proposition a' which yields a false (true) singular judgement whenever a yields a true (resp. false) singular judgement. This is a familiar way of introducing negation in the traditional quantum logic. Łukasiewicz noticed also that

(p. 20) ... *the truth value of any indefinite proposition plus the truth value of its negation, equals 1*, ...

This, when negation is represented by orthocomplementation and truth values are replaced by values of probability measures, is again a generally accepted quantum logic formula.

The notion of the *logical product ab* of indefinite propositions a and b used by Łukasiewicz (1913) seems to arise from the bivalent conjunction applied to singular judgements obtained from a and from b. However, this can be again inferred only from the example studied on p. 19, at the end of which he wrote:

(p. 19) ... *"x is different from 6 and greater than 3"* ... *In algebraic logic, such propositions, connected by the word "and" are called logical products.*

I think, taking into account the way in which Łukasiewicz defined the relation of implication, that the exact definitions of the logical product and the logical sum (which he introduced in the same vague way on p. 20) could be formulated as follows:

The *logical product* of two indefinite propositions a and b is an indefinite proposition ab which yields a true singular judgement if and only if both propositions a and b yield true singular judgements.

The *logical sum* of two indefinite propositions a and b is an indefinite proposition $a + b$ which yields a true singular judgement if and only if at least one of propositions a, b yields a true singular judgement.

We see that contrary to the *relation* of implication, negation, logical product, and logical sum are *operations* on indefinite propositions. However, while the operation of negation is always defined, in the case of logical products and sums we encounter the first difference between Łukasiewicz (1913) original logic dealing with statements containing free variables and his logic adopted to describe experiments on quantum objects.

In the case of the original Łukasiewicz (1913) logic, simultaneous replacing of different free variables by their specific values was always possible, while simultaneous checking of different indefinite propositions which predict the results of quantum experiments sometimes cannot be performed:

Example 2. Let S be a source of photons linearly polarized in the direction x_S, let P_A and P_B be linear polarizers oriented, respectively, in the directions x_A and x_B, and let all three directions x_S, x_A, and x_B be different. If a (b) is an indefinite proposition "photon emitted by S towards P_A (resp. P_B) will pass through it," then neither the logical product, nor the logical sum of a and b can be formed, since the experimental arrangement does not allow us to check them simultaneously.[3]

Therefore, Łukasiewicz' (1913) logic applied to the description of experiments on quantum objects is a partial logic in which logical products and sums are not always defined.

Łukasiewicz defined the equivalence of two indefinite propositions as follows:

(p. 20) . . . *the equivalence* $a = b$ *is identical with the logical product of* $(a < b)(b < a)$ *and means: "from a follows b and from b follows a".*

However, since $a < b$ and $b < a$ are bivalent metalogical statements which are true or not depending on whether the relation of Łukasiewicz implication between a and b holds or not, the equivalence defined by Łukasiewicz is also a relation between indefinite propositions, not an operation on them. Moreover, the word "identical" used by Łukasiewicz obviously means "equivalent in the traditional bivalent logic" or "metalogically equivalent." Therefore, the definition of Łukasiewicz' (1913) relation of equivalence should be symbolically written as follows:

[3] Let us note that if in this example photons are replaced by bullets and polarizers by targets, we obtain a "classical quantum-like situation" of a type studied by Aerts [see Aerts (1981) and many of his subsequent papers]. As far as the property "passing through a target" is concerned, bullets are indivisible objects (a fragment of a bullet does not have this property to the same extent as a whole bullet), which is in contradiction to the paradigm of classical physics (objects are infinitely divisible and their properties are hereditary). Therefore, bullets with respect to this property are "quanta" in the very original sense of this word.

$$(a = b) \Leftrightarrow [(a < b) \ \& \ (b < a)] \tag{2}$$

where $\Leftrightarrow$ and & denote, respectively, classical (bivalent) equivalence and conjunction.

3. THE CALCULUS OF TRUTH VALUES

In order to keep the length of the present paper within reasonable limits, technical details of its last two sections are skipped and given in a forthcoming paper (Pykacz, n.d.-b). In the foregoing quotations from Łukasiewicz (1913) his original symbols are in several formulas replaced in some places by metalogical (bivalent) symbols in order to make the formulas clearer.

3.1. Principles of the Calculus

(p. 21) *The calculus of truth values is based on the following three principles:*

$$I \quad (a = 0) \Leftrightarrow [w(a) = 0] \tag{3}$$

$$II \quad (a = 1) \Leftrightarrow [w(a) = 1] \tag{4}$$

$$III \quad (a < b) \Rightarrow [w(a) + w(a'b) = w(b)] \tag{5}$$

... *Within the calculus of truth values they play the role of axioms.*

0 and *1* in the left-hand sides of metaequivalences I and II denote indefinite propositions which yield, respectively, always false and always true singular judgements. Such indefinite propositions were called by Łukasiewicz (1913) simply *false* and *true*, while the other indefinite propositions were called *neither true nor false*. The metaimplication III expresses what Łukasiewicz called previously *theorem on the truth value of a reason.* Let us note that even in the case of indefinite propositions about results of experiments on quantum objects, due to considerations contained in Section 2.4, the logical product $a'b$ appearing in the consequent of the metaimplication III is defined whenever $a < b$ holds.

3.2. Theorems

Among many results which can be proved on the basis of the adopted definitions and axioms, the following ones deserve special attention:

$$(a' < b) \Rightarrow [w(ab) = w(a) + w(b) - 1] \tag{6}$$

$$(a < b') \Rightarrow [w(a + b) = w(a) + w(b)] \tag{7}$$

Consequents of these metaimplications can be easily recognized as "parts" of the formulas yielding truth values of conjunction and disjunction already studied by Frink (1938):

$$\tau(\text{"}a \text{ and } b\text{"}) = \max[\tau(a) + \tau(b) - 1, 0] \tag{8}$$

$$\tau(\text{"}a \text{ or } b\text{"}) = \min[\tau(a) + \tau(b), 1] \tag{9}$$

Later these formulas were used within the fuzzy set theory by Giles (1976) to define products and sum of fuzzy sets different from the most frequently used Zadeh (1965) operations. Operations on fuzzy sets defined by Giles (1976) were in turn extensively used by the present author (Pykacz, 1990, 1993, 1994) in attempts to define fuzzy quantum logic exclusively with the aid of a triple of operations consisting of intersection, union, and complement for which the De Morgan laws hold. Such a "homogeneous" definition was finally given in (Pykacz, 1994). It should be stressed, however, that the present approach based on Łukasiewicz (1913) is different because of the basic requirement that relations between propositions and operations on them should be defined "operationally," which makes the resulting logic a partial logic.

3.3. The Law of Addition

The following *law of addition*:

(p. 24) $$(ab = 0) \Rightarrow [w(a + b) = w(a) + w(b)] \tag{10}$$

was proved by Łukasiewicz (1913) with the aid of distributivity of his logical product and sum, which in the case of his indefinite propositions containing free variables actually holds. In the case of indefinite propositions about results of experiments on quantum objects distributivity cannot be assured. Nevertheless, the law of addition (10) can be in this case proved due to the following lemma:

$$(ab = 0) \Leftrightarrow (a' < b) \tag{11}$$

The problem becomes more serious if we want to generalize the law of addition to more than two propositions. Łukasiewicz noticed:

(p. 24) *The law of addition can, on the strength of mathematical induction, be extended so as to cover more than two propositions. The following holds:*

$$(\Sigma_{i,j}\, a_i a_j = 0) \Rightarrow [w(\Sigma_i\, a_i) = \Sigma_i\, w(a_i)] \tag{12}$$

$$i \neq j, \qquad i = 1, 2, \ldots, n, \qquad j = 1, 2, \ldots, n$$

Distributivity of the logical product and sum, which cannot be guaranteed for indefinite "quantum" propositions, seems to be essential in passing from

(10) to (12). However, the "generalized law of addition" (12) resembles so much the familiar formulas expressing σ-additivity of probability measures on σ-orthocomplete orthomodular posets that it seems worth adopting it, when it cannot be proved, as an axiom.

4. LINDENBAUM-TARSKI ALGEBRA OF A THEORY WITH ŁUKASIEWICZ LOGIC OF INDEFINITE PROPOSITIONS

The standard construction of the Lindenbaum–Tarski algebra of a theory consists in identifying equivalent propositions and defining partial order with the aid of implication. Meets, joins, and complements should then be generated by operations of conjunction, disjunction, and negation, and they should of course agree with previously defined partial order. Lindenbaum–Tarski algebra of a "classical" theory with Łukasiewicz (1913) logic of indefinite propositions in which logical product and sum of arbitrary propositions is always defined is a Boolean algebra. In the case of "nonclassical" theories satisfying, however, the "generalized law of addition" (12) the resulting Lindenbaum–Tarski algebra is an orthomodular poset which is σ-orthocomplete when in (12) finite sums are replaced by countable ones. Lengthy proofs of these results will be published elsewhere (Pykacz, n.d.-b).

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